Clustering in 1D Binary Elementary Cellular Automata

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ABSTRACT

A variant of cellular automata which allows for changes in cell sizes during their time evolution, is investigated. The variant deals with one-dimensional binary elementary cellular automata. It is shown that these binary one-dimensional cellular automata exclusively form checkerboard patterns. These checkerboard cellular automata form a much smaller class of inequivalent elementary automata than in the non-clustering case.

CCS Concepts

•**Theory of computation → Regular languages;**

Keywords

Cellular automata, clustering

1. INTRODUCTION

Cellular automata (CA) have many well-known variations, such as different cell shapes (for example, hexagonal cells [4]) or probabilistic behaviour [6], amongst others. Another variation is that of adaptable cell sizes, where different homogeneous cells are allowed to merge into one supercell, called a cluster [9]. Clusters in two-dimensional CA have been shown to have advantages in practical applications such as optimisation [8], and the speed-up of ant sorting applications [1].

In this work we consider the behaviour and characteristics of these CA with clustering in the one-dimensional case, and specifically for binary CA. We show that clusters influence the number of inequivalent elementary one-dimensional CA (1DCA), and we give the new inequivalence classification when clustering is applied. Moreover, we show that clustering in binary 1DCA leads to a quick contraction into a stable configuration of checkerboard patterns – we analyse the rate of contraction in these automata.

2. BACKGROUND

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We assume that the reader is familiar with CA, and simply give a summary of the most important definitions in order to establish assumptions and notation. For more detail on the theory or applications of CA, any standard references such as [2, 3, 5, 7, 10] may be consulted.

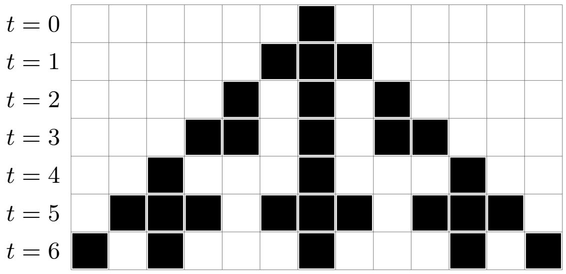
In general, a CA can be an infinite multidimensional array of (possibly different) automata, with a transition rule that governs the simultaneous evolution of the individual cells in the automata array in discrete time steps. In time step $t = 0$, the individual automata are in their respective start states. In this work, only the case where the array is one-dimensional, and bounded (finite) at both ends, is considered. Moreover, it is assumed that the individual cells contain identical automata with only two states each (a binary CA). Hence, the time step evolution of the CA can be graphically represented as a series of tapes with only white or black cells (see Example 1).

In a CA of size n (that is, a CA with n cells), the transition rule f is applied to each cell c_i and a select number of neighbouring cells, to obtain the state of c_i in the next time step. The collection of neighbouring cells (called the neighbourhood) has distance $r = 1$ if each of the neighbours is geometrically adjacent to cell c_i. In the one-dimensional case, the neighbourhood of cell c_i with distance $r = 1$ is simply the set of cells $\{c_{i-1}, c_{i+1}\}$ or the set $\{c_{i-1}, c_i, c_{i+1}\}$. The CA with the latter neighbourhood are called *elementary* CA. For the elementary binary 1DCA, there are only 2^{2^3} possible transition rules [10]. These 256 transition rules can be ordered into equivalence classes, so that there are only 88 inequivalent possiblities. We pinpoint the number of inequivalent elementary CA in the case where clustering is allowed.

In a bounded array, the transition rule has to be defined at the limits of the array. In a *null-bounded* CA, the transition rule assumes that the leftmost cell (c_0) has a left neighbour with a null value (in the binary case, null is zero). Similarly, the rightmost cell (c_{n-1}) has a right neighbour with a null value. On the other hand, one may also consider a *periodic* CA, where the array is seen as a ring, so that the left neighbour of cell c_0 is cell c_{n-1} and the right neighbour of cell c_{n-1} is cell c_0.

EXAMPLE 1. *Consider a 1DCA with transition rule 150 (see [10] for the numbering of rules of elementary CA), and an initial state where all* $c_i = 0$*, for* $i \in \{0, 1, \ldots, 5, 7, \ldots, 13\}$*, and* $c_6 = 1$*. Rule 150 is defined as*

$$f_{c_i}(t+1) = c_{i-1}(t)\ XOR\ c_i(t)\ XOR\ c_{i+1}(t).$$



If one indicates 0 *by a white square, and* 1 *by a black square, it is easy to see how the application of the transition function leads to the state of the array in each subsequent time step. For example, from time step* $t = 0$ *to* $t = 1$*, the transition function is applied to a neighbourhood* 000 *for cells* c_i *when* $i \in \{0, 1, 2, 3, 4, 8, 9, 10, 11, 12\}$*, and since* $f(000) = 0$*, those cells have the value* 0 *again in time step* $t = 1$*. On the other hand, for cell* 5*, one finds that* $f(001) = 1$*, and hence cell* 5 *has the value* 1 *in time step* $t = 1$*. Similarly, for cell* 6 *and* 7*, it holds that* $f(010) = 1$ *and* $f(100) = 1$*, and therefore cells* 6 *and* 7 *also have the value* 1 *in time step* $t = 1$*.*

3. ONE-DIMENSIONAL BINARY CA WITH CLUSTERING

Clustering represents the idea that adjacent cells with the same value merge into one larger cell with that value.

Definition 1. Let C be a one-dimensional binary CA of size n. Let $0 \leq i, j \leq n - 1$. Two cells c_i and c_j are adjacent if $|i - j| = 1$.

A cluster B in C is a set of adjacent cells from C such that all cells in B have the same value.

Definition 2. Let C be a one-dimensional CA of size n. Let $0 \leq i \leq j \leq n - 1$, and consider the cells $c_i, c_{i+1}, \ldots, c_j$. If $c_i = c_{i+1} = \ldots = c_j$, then the set $B = \{c_i, c_{i+1}, \ldots, c_j\}$ is a cluster in C. The cluster B has size $j - i + 1$.

Fig. 1 shows five clusters. The first cluster has size three and value 0, the second has size one and value 1, the third has size two and value 0, the fourth again has size one and value 1, and the last has size two and value 0. For clarity, the diagram indicates the original cells that were merged to form the cluster.

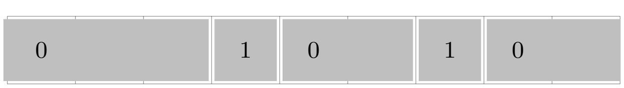


Figure 1: Clusters in a 1DCA

In a CA without clustering, the transition rule is applied to calculate the state of each cell of the CA in the next time step. When clustering is allowed, the transition rule is applied and additionally, all adjacent cells with the same value are merged into one cluster. This implies that (apart from time step 0), any two adjacent clusters in a 1D binary CA must have different values. Hence, the only possible patterns are of the form $0101\ldots01$, $0101\ldots010$, $1010\ldots10$, $1010\ldots101$, or the two degenerative cases with a single cluster containing a 0 or a 1. Without loss of generality, we may assume that only the above six patterns are used as initial states of the CA (if a different initial state occurs, the merging step into clusters can immediately be applied to form one of the six patterns).

The reader may note that, in the case of periodic boundaries, the pattern $0101\ldots010$ is equivalent to the pattern $0101\ldots01$. Similarly, $1010\ldots101$ is equivalent to $1010\ldots10$. In the case of a CA with a single cluster, the left and right neighbours have the same value as the cluster itself. In the null boundary case, the single cluster has a null left neighbour and null right neighbour.

We now investigate the formation and properties of these basic patterns first under periodic and then under null boundary conditions.

4. PERIODIC BOUNDARY CONDITIONS

In this section, we consider n-cluster 1D binary CA with clustering under periodic boundary conditions (1DCA-PCB).

THEOREM 1. *Any 1DCA-PBC has an even number of clusters, for all* $n > 1$.

PROOF. Consider any 1DCA-PCB $\mathcal{C}$ with clusters

$$c_0, c_1, \ldots, c_{n-1}.$$

For any cluster c_i with $0 \leq i < n - 1$ it must hold that $c_i \neq c_{i+1}$, and thus the clusters must form a pattern $xyxy\ldots$, with $x, y \in \{0, 1\}$. Suppose $n > 1$ and n is odd. Then the pattern has the form $xyxy\ldots x$. But under periodic boundary conditions, the left neighbour of c_0 is c_{n-1}, which implies that $c_0 = c_{n-1} = x$, and hence c_0 and c_{n-1} have the same value and must merge into a cluster, reducing the number of clusters in $\mathcal{C}$ by one. The result follows. □

Definition 3. The merging of cells in a cluster from time step t to time step $t + 1$, changing the number of clusters from n_t to n_{t+1}, is known as a contraction with value $\frac{n_t}{n_{t+1}}$.

If the contraction has value one, then no clustering has taken place from time step t to time step $t + 1$. If the contraction has a value larger than one, then clustering has occurred by moving from time step t to $t + 1$, and hence the CA has fewer clusters in time step $t + 1$ than in time step t. The largest possible contraction takes place if the value of the contraction from time t to $t + 1$ is n_t.

Definition 4. Let $\mathcal{C}$ be a 1DCA-PBC. Let the value of the contraction from time t to $t + 1$ be given by $k = \frac{n_t}{n_{t+1}}$. If $k = n_t$, the contraction is a maximal contraction.

The patterns for any 1DCA-PBC through its time evolution can be easily described. Consider $\mathcal{C}$ with transition rule f at any time t_j. Suppose the current cluster pattern is $abab\ldots ab$, with $a, b \in \{0, 1\}$. Then f is applied to neighbourhoods of the form aba and bab (see Fig. 2). Consider two adjoining clusters c_i and c_{i+1}. Without loss of generality, suppose that c_i has value a and c_{i+1} has value b. Then $f_{c_i}(t + 1)$ is applied to a neighbourhood of the form aba, and $f_{c_{i+1}}(t + 1)$ to a neighbourhood of the form bab. For notational convenience, and where the specific values of c_i and t are not important, we simply write $f(aba)$ to mean $f_{c_i}(t + 1)$, where c_i has the value b, and both its neighbours have the value a.

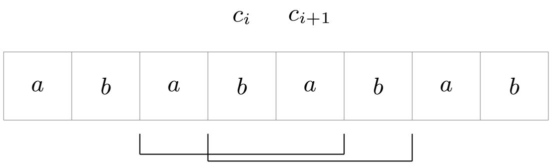


Figure 2: Neighbourhoods in a 1DCA

THEOREM 2. *Let $\mathcal{C}$ be a 1DCA-PBC with transition rule f. Then a maximal contraction in one time step is possible iff $f_{c_i}(t+1) = f_{c_{i+1}}(t+1)$, for all i such that $0 \leq i < n$.*

PROOF. Suppose a maximal contraction occurs from time step t to time step $t+1$. If there are n_t clusters in time step t and n_{t+1} clusters in time step $t+1$, then by Definition 4 it holds that $n_{t+1} = 1$. But if there is only one cluster in time step $t+1$, this means that the transition from time t resulted in all clusters having the same value in time step $t+1$. That is, $f_{c_i}(t+1) = f_{c_{i+1}}(t+1)$, for all i such that $0 \leq i < n$.

For the converse, assume that $f_{c_i}(t+1) = f_{c_{i+1}}(t+1)$, for all i such that $0 \leq i < n$. Hence, $f(aba) = x$ and $f(bab) = y$, and $x = y$. If $x = y$, then $f(abab \ldots ab) = x \ldots x$ at time step $t+1$. Hence, a single cluster of size 1 is formed, and a maximal contraction occurred in one step. $\square$

On the other hand, it may be possible that no contraction occurs.

THEOREM 3. *Let $\mathcal{C}$ be a 1DCA-PCB with transition rule f. Then no contraction occurs in time step t to $t+1$ iff $f_{c_i}(t+1) \neq f_{c_{i+1}}(t+1)$, for all i such that $0 \leq i < n$.*

PROOF. Analogous to the proof of Theorem 2:

Suppose no contraction occurs from time step t to time step $t+1$. If there are n_t clusters in time step t and n_{t+1} clusters in time step $t+1$, then by Definition 4 it holds that $n_t = n_{t+1}$. But if there are the same number of clusters in both time step t and $t+1$, this means that the transition from time t resulted in all adjoining clusters having different values in time step $t+1$. That is, $f_{c_i}(t+1) \neq f_{c_{i+1}}(t+1)$, for all i such that $0 \leq i < n$.

For the converse, assume that $f_{c_i}(t+1) \neq f_{c_{i+1}}(t+1)$, for all i such that $0 \leq i < n$. Hence, $f(aba) = x$ and $f(bab) = y$, and $x \neq y$. If $x \neq y$, then $f(abab \ldots ab) = xyxy \ldots y$ at time step $t+1$. Hence, no contraction occurred. $\square$

Note that a 1DCA-PBC exhibits only either a maximal contraction or no contraction in a given time step – no other values are possible for contractions. This is different from the null boundary case, as discussed in Sect. 5.

THEOREM 4. *A 1DCA-PBC exhibits either a maximal contraction or no contraction in a given time step.*

PROOF. By contradiction. Assume a contraction with value $k = \frac{n_t}{n_{t+1}}$ occurs, with $1 < k < n_t$. Since $k > 1$, it must hold that $n_t > 1$. If $n_t = 2$, either no contraction occurs, or a contraction occurs from two clusters to one, and hence is a maximal contraction. Assume $n_t = 2p$, for some $p \geq 2$ (by Theorem 1, n_t must be even), and $n_t > 2$. Consider any two adjoining clusters $c_i = a$ and $c_{i+1} = b$ for some $0 \leq i < n_t$. If $f(aba) = x$ and $f(bab) = y$, then either $x = y$ or $x \neq y$. If $x = y$, then by Theorem 2 a maximal contraction occurs. If $x \neq y$, then by Theorem 3, no contraction occurs. $\square$

We now consider the possible patterns that can occur during the time evolution of a 1DCA-PBC.

LEMMA 1. *For any 1DCA-PBC, if*

$$f(abab \ldots ab) = baba \ldots ba,$$

then

$$f(baba \ldots ba) = abab \ldots ab,$$

for $a, b \in \{0, 1\}$.

PROOF. Consider the current state of the CA, given as $abab \ldots ab$. For clarity, index each element so that the current state is given by $a_0 b_1 a_2 b_3 \ldots a_{n-2} b_{n-1}$. Then $a_0(t+1) = f(b_{n-1} a_0 b_1) = b$, and $b_1(t+1) = f(a_0 b_1 a_2) = a$. In general, $f(b_{i-1} a_i b_{i+1}) = b$ and $f(a_{i-1} b_i a_{i+1}) = a$, for $0 < i < n-1$. The result follows directly. $\square$

Similarly,

LEMMA 2. *For any 1DCA-PBC, if*

$$f(abab \ldots ab) = abab \ldots ab,$$

then

$$f(baba \ldots ba) = baba \ldots ba,$$

for $a, b \in \{0, 1\}$.

PROOF. Consider the current state of the CA, given as $abab \ldots ab$. For clarity, index each element so that the current state is given by $a_0 b_1 a_2 b_3 \ldots a_{n-2} b_{n-1}$. Then $a_0(t+1) = (b_{n-1} a_0 b_1) = a$, and $b_1(t+1) = f(a_0 b_1 a_2) = b$. In general, $f(b_{i-1} a_i b_{i+1}) = a$ and $f(a_{i-1} b_i a_{i+1}) = b$, for $0 < i < n-1$. The result follows directly. $\square$

The patterns $abab \ldots$ and $baba \ldots$ are known as dual patterns.

THEOREM 5. *Let $\mathcal{C}$ and $\mathcal{C}'$ be 1DCA-PBCs with the same transition rule f. Let $\mathcal{C}$ have the initial state $0101 \ldots 01$ and $\mathcal{C}'$ have the initial state $1010 \ldots 10$. Then $\mathcal{C}$ and $\mathcal{C}'$ show dual patterns during time evolution if no contraction takes place.*

PROOF. Consider $\mathcal{C}$. Since no contraction takes place, the initial state $0101 \ldots 01$ can only produce cyclic repetitive states of the form (i) $0101 \ldots 01$ to $1010 \ldots 10$, or (ii) $0101 \ldots 01$ to $0101 \ldots 01$. The result follows from Lemmas 1 and 2. $\square$

This dual behaviour does not hold in the case where the first time step leads to a maximal contraction.

THEOREM 6. *Let $\mathcal{C}$ and $\mathcal{C}'$ be 1DCA-PBCs with the same transition rule f. Let $\mathcal{C}$ have the initial state $0101 \ldots 01$ and $\mathcal{C}'$ have the initial state $1010 \ldots 10$, and f leads to a maximal contraction in time step $t = 1$. Then $\mathcal{C}$ and $\mathcal{C}'$ show the same pattern development.*

PROOF. Since maximal contraction takes place, the initial states $0101 \ldots 01$ or $1010 \ldots 10$ produce a pattern 0 or a pattern 1 in time step $t = 1$. Consider a maximal contraction to 0 (resp. 1). This occurs if $f(010) = 0$ (resp. 1) and

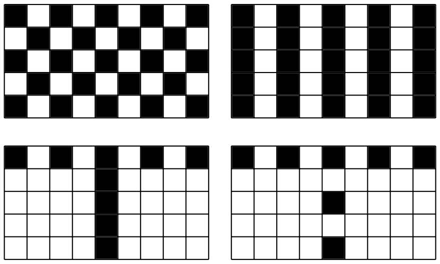


Figure 3: The elementary 1DCA-PCB checkerboard patterns

$f(101) = 0$ (resp. 1). Therefore, the initial states of both $\mathcal{C}$ and $\mathcal{C}'$ contract to 0 (resp. 1). Then, since $\mathcal{C}$ and $\mathcal{C}'$ have the same transition rule f, the subsequent time steps are represented by $f(000)$ (resp. $f(111)$), and hence are the same for each time step. The result follows. □

Under maximal contraction, we note that the transition rule f on the neighbourhood of size three can therefore only have the following possibilities:

- 000 leads to 0 or 1; and
- 111 leads to 0 or 1.

Under no contraction, the possibilities are

- 010 leads to 010 or 101; and
- 101 leads to 010 or 101.

Hence,

COROLLARY 1. *In a 1DCA-PBC, the application of the transition rule can exhibit the following patterns, where $a, b \in \{0, 1\}$:*

- *$abab \dots ab$ alternates with $baba \dots ba$*
- *$abab \dots ab$ occurs in each time step*
- *$abab \dots ab$ contracts to a, and a repeats*
- *$abab \dots ab$ contracts to b, and b repeats*
- *$abab \dots ab$ contracts to a, which alternates with b*
- *$abab \dots ab$ contracts to b, which alternates with a*
- *$abab \dots ab$ contracts to a, followed by a repeating b*
- *$abab \dots ab$ contracts to b, followed by a repeating a.*

These patterns are illustrated in Fig. 3. Note the checkerboard effect.

Binary 1DCA (without clustering) with neighbourhood size three has $2^{2^3} = 256$ unique transition rules. These are known as the elementary CA. A large number of these auomata are equivalent up to a simple transformation (reflection and complement), and there are only 88 unique elementary CA. The question arises how many elementary 1DCA are unique in the case of clustering.

THEOREM 7. *There are exactly four elementary 1DCA-PBC with clustering.*

PROOF. From Corollary 1, excluding reflection and complement, it follows directly that there are only four unique possibilities. □

5. NULL BOUNDARY 1DCA

Pattern development for 1DCA under null boundary conditions (1DCA-NBC) differs from that in 1DCA-PBC, and is often more interesting than the (somewhat restricted) patterns found with 1DCA-PCB.

In the null boundary case, the zeroes to the left of the first cluster and to the right of the last cluster, do not form part of the cluster, and are simply there as values for the transition rule calculation. It is therefore possible that the transition rule for cluster c_0 can be applied to the pattern $0ab$, where $a, b \in \{0, 1\}$. Similarly, for cluster c_{n-1}, the rule can be applied to $xy0$, where $x, y \in \{0, 1\}$. Depending on the specific transition rule, this may lead to contraction at the ends of the array only. In contrast to the periodic bound case, the 1DCA-NBC may have either an even or an odd number of clusters. For that reason, we write patterns as $ab \dots xy$ when the number of clusters is even or odd, but $ab \dots ab$ when the number of clusters is even.

LEMMA 3. *An n-cluster 1DCA-PBC $\mathcal{C}$ and an n-cluster 1DCA-NBC $\mathcal{C}'$ with the same transition rule and the initial state $ab \dots ab$ show the same pattern development if $f(0ab) = f(bab)$ and $f(ab0) = f(aba)$.*

PROOF. Immediate. An example is the rule

$$f(c_i(t+1)) = \text{NOT}(c_{i-1}(t) \text{ OR } c_i(t) \text{ OR } c_{i+1}(t))$$

on the initial state $1010 \dots 10$ (rule number 1). □

LEMMA 4. *An n-cluster 1DCA-PBC $\mathcal{C}$ and an n-cluster 1DCA-NBC $\mathcal{C}'$ with the same transition rule and the initial state $ab \dots ab$ show different pattern development if $f(0ab) \neq f(bab)$ or $f(ab0) \neq f(aba)$.*

PROOF. Immediate. An example is the rule

$$\begin{aligned} f(c_i(t+1)) &= (c_{i-1}(t) \text{ XOR } c_i(t) \text{ XOR } c_{i+1}(t)) \\ &\quad \text{AND NOT}(c_i(t)) \end{aligned}$$

on the initial state $1010 \dots 10$ (rule number 18). □

1DCA-PBC can only show no contraction, or maximal contraction in one time step. This is not the case for 1DCA-NBC, where stepwise contraction on one side of the array or stepwise contraction on both sides of the array are possible. The lemmas below capture the conditions under which such single step contractions occur.

LEMMA 5. *For a 1DCA-NBC, a non-maximal contraction of $\frac{n_t}{n_t - 1}$ is possible.*

PROOF. A contraction with value $\frac{n_t}{n_t-1}$ occurs when there is one less cluster in the CA at time $t+1$, compared to time t. This can happen only if there is a single contraction at either the left or right end of the CA array, or once in the middle. First suppose that the 1DCA-NBC at time t has the state $ab \dots ab$.Consider the left side of the CA. If $f(0ab) = f(aba)$, but $f(aba) \neq f(bab)$, then at time step $t+1$ a contraction occurs on the leftmost two clusters. Since $f(aba) \neq f(bab)$, no contraction can occur in the middle of the CA array. Consider the right side of the CA. If $f(ab0) \neq f(bab)$, no contraction occurs. The above conditions are all non-conflicting, and hence a non-maximal contraction of $\frac{n_t}{n_t-1}$ is possible.

For the second possible CA state $ab \dots ba$, an analogous argument holds, under the condition that $f(0ab) = f(aba)$, $f(aba) \neq f(bab)$, and $f(ba0) \neq f(aba)$.

□

A non-maximal contraction at both sides of the CA is also possible.

COROLLARY 2. *For a 1DCA-NBC, a non-maximal contraction of* $\frac{n_t}{n_t-2}$ *is possible.*

PROOF. As in the proof of Lemma 5 above, but with conditions $f(0ab) = f(aba)$, $f(aba) \neq f(bab)$ and $f(ab0) = f(bab)$ for the state $ab \ldots ab$. For the state $ab \ldots ba$, the conditions are $f(0ab) = f(aba)$, $f(aba) \neq f(bab)$ and $f(ba0) = f(aba)$. □

LEMMA 6. *A 1DCA-NBC* $\mathcal{C}$ *with an odd number of clusters shows maximal contraction in one time step iff* $f(0ab) = f(aba) = f(bab) = f(ba0)$.

PROOF. If $\mathcal{C}$ has an odd number n_t of clusters, it must have an initial state of the form $ab \ldots aba$. If $f(0ab) = f(aba) = f(bab) = f(ba0)$, then the first time step produces a single cluster with value a or b, which constitutes a maximal contraction.

For the converse, suppose $\mathcal{C}$ is any 1DCA-NBC with a maximal contraction $k = \frac{n_t}{n_{t+1}}$ between time steps n_t and n_{t+1}. Then $n_{t+1} = 1$, which implies that the CA at time step $t+1$ is a single cluster with one value (a or b). That is possible only if $f(0ab) = f(aba) = f(bab) = f(ba0)$. □

LEMMA 7. *A 1DCA-NBC* $\mathcal{C}$ *with an even number of clusters shows maximal contraction in one time step iff* $f(0ab) = f(aba) = f(bab) = f(ab0)$.

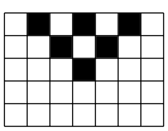
PROOF. Analogous to Lemma 6 above. □

Example 2 illustrates the non-maximal contractions in a 1DCA-NBC.

EXAMPLE 2. *Let* $\mathcal{C}$ *be a 7-cluster 1DCA-NBC with*

$$f_{c_i}(t+1) = c_{i-1}(t) \text{ AND } (\text{NOT } c_i(t)) \text{ AND } c_{i+1}(t)$$

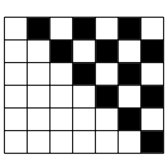
(known as rule 32). Then $\mathcal{C}$ *contracts with* $\frac{n_t}{n_t-2}$ *given initial state* 0101010*. This follows, as* $f(001) = f(010) = 0$, $f(010) \neq f(101)$, *and* $f(100) = f(010) = 0$*. Note that* $\mathcal{C}$ *stabilizes within three time steps to an all-zero CA.*



On the other hand, consider a 1DCA-NBC $\mathcal{C}'$*, where the transition rule* f *is given by*

$$f_{c_i}(t+1) = c_{i-1}(t) \text{ AND } (\text{NOT } c_i(t))$$

(also known as rule 48). With the same initial state 0101010*, there is only a left-sided contraction, since* $f(001) = f(010) = 0$, $f(010) \neq f(101)$, *and* $f(100) \neq f(010)$*. Here* $\mathcal{C}'$ *takes five steps to stabilize, to a CA with only one non-zero cluster.*



Other almost-maximal contractions occur when the initial state contracts to two clusters or three clusters in one time step.

LEMMA 8. *Consider any 1DCA-NBC* $\mathcal{C}$ *with an even number of clusters and initial state* $ab \ldots ab$*, such that the transition rule forms two disjoint partitions* $f(aba) = f(bab) = x$ *and* $f(001) = f(100) = y$*, with* $x \neq y$*. If* $f(0ab) \neq f(ba0)$*, then* $\mathcal{C}$ *contracts in time step* $t = 1$ *to* ba*, with a contraction value* $\frac{n_t}{2}$.

PROOF. Since $f(aba) = f(bab)$, all clusters c_i, with $0 < i < n-1$, will contract from time step $t = 0$ to time step $t = 1$ to a single value (say x). Clusters c_0 and c_{n-1} in time step $t = 1$ must have different values, and therefore $\mathcal{C}$ contracts within the first time step to yxx or xxy.

The result follows. □

LEMMA 9. *Consider any 1DCA-NBC* $\mathcal{C}$ *with an odd number of clusters and initial state* $ab \ldots aba$*, such that the transition rule forms two disjoint partitions* $f(aba) = f(bab) = x$ *and* $f(001) = f(100) = y$*, with* $x \neq y$*. If* $f(0ab) = f(ba0)$*, then* $\mathcal{C}$ *contracts in time step* $t = 1$ *to* bab*, with a contraction value* $\frac{n_t}{3}$.

PROOF. Since $f(aba) = f(bab)$, all clusters c_i, with $0 < i < n-1$, will contract from time step $t = 0$ to time step $t = 1$ to a single value (say x). Clusters c_0 and c_{n-1} in time step $t = 1$ must have the same value, say y, so that $\mathcal{C}$ contracts within the first time step to yxy. Suppose $x = y$. Then it must hold that $f(0ab) = f(aba)$, but for $a = 0$ this means that $f(001) = f(010)$, which is false.

The result follows. □

LEMMA 10. *For any 1DCA-NBC, suppose that*

$$f(abab \ldots ab) = baba \ldots ba.$$

Then

$$f(baba \ldots ba) = abab \ldots ab,$$

for $a, b \in \{0, 1\}$*, if* $f(aba) = a$, $f(bab) = b$, $f(ba0) = b$, $f(0ab) = b$, $f(ab0) = a$ *and* $f(0ba) = a$.

PROOF. Consider the current state $abab \ldots ab$. For clarity, index each element so that the current state is given by $a_0b_1a_2b_3 \ldots a_{n-2}b_{n-1}$. Then $a_0(t+1) = f(0a_0b_1) = b$, and $b_1(t+1) = f(a_0b_1a_2) = a$. In general, $f(b_{i-1}a_ib_{i+1}) = b$ and $f(a_{i-1}b_ia_{i+1}) = a$, for $0 < i < n-1$. The four conditions on the left and right boundaries ensure the dual behaviour at every step for $i = 0$ and $i = n-1$. The result follows. □

COROLLARY 3. *For any given 1DCA-NBC, suppose that*

$$f(abab \ldots aba) = baba \ldots bab.$$

Then it holds that

$$f(baba \ldots bab) = abab \ldots aba,$$

for $a, b \in \{0, 1\}$*, if* $f(aba) = a$, $f(bab) = b$, $f(ba0) = b$, $f(0ab) = b$, $f(ab0) = a$ *and* $f(0ba) = a$.

Similarly,

LEMMA 11. *For any 1DCA-NBC, suppose that*

$$f(abab \ldots ab) = abab \ldots ab.$$

Then

$$f(baba\dots ba) = baba\dots ba,$$

for $a, b \in \{0,1\}$*, if* $f(aba) = b$*,* $f(bab) = a$*,* $f(ba0) = a$*,* $f(0ab) = a$*,* $f(ab0) = b$ *and* $f(0ba) = b$*.*

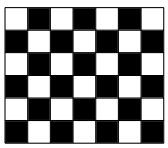
Proof. Consider the current state $abab\dots ab$. For clarity, index each element so that the current state is given by $a_0b_1a_2b_3\dots a_{n-2}b_{n-1}$. Then $a_0(t+1) = (b_{n-1}a_0b_1) = a$, and $b_1(t+1) = f(a_0b_1a_2) = b$. In general, $f(b_{i-1}a_ib_{i+1}) = a$ and $f(a_{i-1}b_ia_{i+1}) = b$, for $0 < i < n-1$. The result follows directly. □

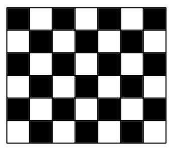
Again, the result holds for an odd number of clusters as well.

Example 3. *In rule 50, given by*

$$(P\ OR\ Q\ OR\ R)\ XOR\ Q,$$

the conditions hold for Lemma 10, and the checkerboard pattern alternates:





An example of a rule where Lemma 11 holds, is

$$f_{c_i}(t+1) = (NOT(c_{i-1}(t)\ OR\ c_{i+1}(t))\ AND\ c_i(t)$$

(rule 4). In this case, $f(010) = 1$*,* $f(101) = 0$*,* $f(100) = 0$*,* $f(001) = 0$*,* $f(010) = 1$ *and* $f(010) = 1$*.*

In 1DCA-PBC, alternating patterns of a single 0 and single 1 may occur. In the case of 1DCA-NBC, it is also possible that an alternating 01 and 10 pattern occurs.

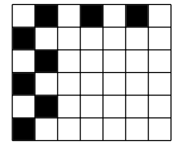
Lemma 12. *For any 1DCA-NBC* $\mathcal{C}$ *with an even number of clusters and initial state* $ab\dots ab$*, suppose that the transition rule forms two disjoint partitions* $f(aba) = f(bab) = x$ *and* $f(001) = f(100) = y$*, with* $x \neq y$*. If* $f(0ab) \neq f(ab0)$*, then* $\mathcal{C}$ *contracts in time step* $t = 1$ *to* ba*, which repeatedly alternates with* ab *for all subsequent time steps.*

Proof. Since $f(aba) = f(bab)$, all clusters c_i, with $0 < i < n-1$, will contract from time step $t = 0$ to time step $t = 1$ to a single value (say x). If $a = 0$, then $f(0ab) = f(001) = y$ and hence cluster c_0 gets value y, where $y \neq x$. At the same time, cluster c_{n-1} gets value x, because $f(ab0) = f(aba) = x$. Hence, in time step 1, $\mathcal{C}$ has state $yxx = yx$. In the next time step, $f(yx) = xy$, since $f(yx) = f(0yx)$ followed by $f(yx0)$, and $f(0ab) \neq f(ba0)$. Similarly, if $a = 1$, then $f(0ab) = f(bab) = x$ and hence cluster c_0 gets value x. At the same time, cluster c_{n-1} gets value y, because $f(ab0) = f(100) = y$. Hence, in time step $t = 1$, $\mathcal{C}$ has state $xxy = xy$. In the next time step, $f(xy) = yx$, since $f(xy) = f(0xy)$ followed by $f(xy0)$, and $f(0ab) \neq f(ba0)$.
The result follows. □

Example 4. *Rule 82 is an example of a rule where the alternating patterns* 01 *and* 10 *occur:*

$$\begin{aligned} f_{c_i}(t+1) &= (c_{i-1}(t)\ OR\ (c_i(t)\ AND\ c_{i+1}(t))) \\ &\quad XOR\ c_{i+1}(t). \end{aligned}$$

Note that this rule causes an almost-maximal contraction to two alternating clusters, as shown below.



However, Lemma 12 does not hold for a 1DCA-NBC with an odd number of clusters. For the case that the number of clusters is odd, we have:

Lemma 13. *For any 1DCA-NBC* $\mathcal{C}$ *with an odd number of clusters and initial state* $ab\dots ba$*, suppose that the transition rule forms two disjoint partitions* $f(aba) = f(bab) = x$ *and* $f(001) = f(100) = y$*, with* $x \neq y$*. If* $f(0ab) = f(ba0)$*, then* $\mathcal{C}$ *contracts within three time steps to a single cluster or to three clusters, which repeat for all subsequent time steps.*

Proof. Since $f(aba) = f(bab)$, all clusters c_i, with $0 < i < n-1$, will contract from time step 0 to time step $t = 1$ to a single value (say x). If $a = 0$, then $f(0ab) = f(001) = y$ and hence cluster c_0 gets value y in time step $t = 1$. But $f(ba0) = f(0ab) = y$, which means that cluster c_{n-1} gets the value y. Hence, in time step 1, $\mathcal{C}$ has state yxy. But now $x \neq y$ and therefore $f(yxy)$ is given by $f(0yx)$, followed by $f(yxy)$, followed by $f(xy0)$. But $f(0yx) = f(xy0)$ and $y \neq x$. Hence, from time step $t = 2$ onwards, if $f(0yx) \neq f(yxy)$, then $f(yxy) = yxy$, or, if $f(0xy) = f(yxy)$, then $f(yxy) = z$, where $z \in \{a, b\}$.
The result follows. □

It is now possible to quantify the number of elementary 1DCA-NBC. We use the terminology *contraction pattern* to denote the effect of subsequent applications of the transition function. For example, if $f(ab) = ba$ and $f(ba) = ab$, then the contraction pattern is alternating. Or, if $f(ab) = ab$ and $f(ba) = ba$), then the contraction pattern is repeating.

From the lemmas above, in a 1DCA-NBC, the following patterns occur:

- the four unique patterns of the 1DCA-PBC;
- a stepwise single-sided non-maximal contraction which repeats the contraction pattern;
- a stepwise single-sided non-maximal contraction which alternates the contraction pattern;
- stepwise a double-sided non-maximal contraction which repeats the contraction pattern;
- stepwise a double-sided non-maximal contraction which alternates the contraction pattern;
- a maximal contraction to one symbol which repeats, or almost-maximal contraction to two symbols which repeat;
- a maximal contraction to one symbol which alternates, or almost-maximal contraction to two symbols which alternate;
- an almost-maximal contraction to three symbols that repeat;
- a stepwise single-sided almost-maximal contraction to three symbols that repeat; and
- a stepwise double-sided almost-maximal contraction to three symbols that repeat.

With reflection and complement taken into account, there are 15 elementary 1DCA-NBC.

6. CONCLUSION

Under periodic boundary conditions, 1D CA with clustering show only four unique elementary automata. Under null boundary conditions, the number of elementary automata increases to 15.

Our next step is to investigate asynchronous 1DCA with clustering, to find the number of elementary CA in this case.

7. ACKNOWLEDGEMENT

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